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Linear Difference and Differential Equations.

By Tomlinson Fort.

Let all the coefficients of a linear difference or differential equation have the same period ω; then if $y(x)$ is a solution of the equation $y(x+\omega)$ is also a solution. In Part I of this paper I set up equations where this fact is used as a basis for generalization. I treat equations of the second order and develop only a few fundamental facts and these primarily with their application to Part III of the paper in view although I hope that Part I will not be without interest on its own account.

Part II is minor in importance and is largely preparatory to Part III. In Part III I take up the self-adjoint boundary-value problems in one dimension. In the case of the differential equation in addition to generalization a number of new facts are brought out relative to the characteristic values for the problem. The method employed is applicable with little change to the difference equation. The results here are essentially the same as for the differential equation and so far as I know wholly new.

When a common notation does not seem practical the discussion for the difference equation is given in greater detail.

PART I.

§ 1. DEFINITION OF THE COEFFICIENTS.

(a) *The Difference Equation.*

Consider the difference equation

$$L(i)y(i+1)+M(i)y(i)+N(i)y(i-1)=0, \tag{1}$$

where the independent variable i is restricted to integral values, $L(i)$, $M(i)$ and $N(i)$ defined when

$$a+1\leqq i\leqq a+\omega \tag{2}$$

and $L(i)\,N(i)\neq 0$ at any point.

Let $a_{11}(i)$ and $a_{12}(i)$ be functions defined when $a\leqq i\leqq a+\omega+1$ and subject only to the restrictions that if $a_{12}(a)\neq 0$, $a_{11}(a+1)$ and $a_{12}(a+1)$ be not both zero and that if $a_{12}(a)=0$, $a_{11}(a)\neq 0$ and $a_{11}(a+1)\neq 0$. Let $b_{11}(i)$ and $b_{12}(i)$

be functions defined when $a+\omega\leqq i\leqq a+2\omega+1$ and subject only to the restrictions that they do not both vanish at the same point and that either $b_{11}(i)$ does not vanish at all or vanishes identically. We let

$$\left.\begin{aligned} u_n(i)&=a_{11}(i)y_n(i)+a_{12}(i)y_n(i-1),\\ v_n(i)&=b_{11}(i)y_n(i+1)+b_{12}(i)y_n(i).\end{aligned}\right\}\tag{3}$$

It is desired that L, M and N be so defined, not all identically zero, when $i=a$ *and when* $a+\omega+1\leqq i\leqq a+2\omega+1$ *that if* y_n *is any solution of* (1), *when* $a\leqq i\leqq a+\omega+1$,

$$u_n(i)\equiv v_{\bar n}(i+\omega),\tag{4}$$

where $y_{\bar n}$ *is a solution of* (1) *also.*

Let y_1 and y_2 be two linearly independent solutions of (1) and adopt the notation $(r,s;\,p,q)\equiv r(p)s(q)-r(q)s(p)$. Regard

$$u_j(i)=v_{\bar j}(i+\omega),\qquad j=1,2,\tag{5}$$

as difference equations in $y_{\bar 1}(i+\omega)$ and $y_{\bar 2}(i+\omega)$, respectively. They are clearly solvable. We desire, moreover, that $y_{\bar 1}(i+\omega)$ and $y_{\bar 2}(i+\omega)$ be linearly independent. To this end define first $L(a)$, $M(a)$ and $N(a)$ so that $u_1(i)$ and $u_2(i)$ are linearly independent. This will be the case if

$$(u_1,u_2;\,a+1,a)\tag{6}$$

be different from zero.

If $a_{12}(a)\neq 0$, let a_{21} and a_{22} be two numbers such that

$$a_{11}(a+1)a_{22}-a_{12}(a+1)a_{21}=A\neq 0,$$

and let

$$\frac{M(a)}{N(a)}=\frac{a_{11}(a)-a_{22}}{a_{12}(a)},\quad \frac{L(a)}{N(a)}=-\frac{a_{21}}{a_{12}(a)}.$$

Then using the difference equation, (6) reduces to

$$A\big(y_1(a+1)y_2(a)-y_2(a+1)y_1(a)\big),$$

which is different from zero. If $a_{12}(a)=0$, no definition of $L(a)$, $M(a)$ and $N(a)$ proves necessary.

When $b_{11}\neq 0$ $y_{\bar 1}(a+\omega)$ and $y_{\bar 2}(a+\omega)$ are arbitrary. Let b_{21} and b_{22} be two numbers such that $b_{11}(a+\omega)b_{22}-b_{21}b_{12}(a+\omega)=B\neq 0$, and let

$$\begin{aligned} y_{\bar 1}(a+\omega)&=b_{11}(a+\omega)u_1(a+1)-b_{21}u_1(a),\\ y_{\bar 2}(a+\omega)&=b_{11}(a+\omega)u_2(a+1)-b_{21}u_2(a),\end{aligned}$$

whereupon by the use of equations (5)

$$\begin{aligned} \frac{M(a+\omega+1)}{L(a+\omega+1)}&=-\frac{(y_{\bar 1},y_{\bar 2};\,a+\omega+2,a+\omega)}{(y_{\bar 1},y_{\bar 2};\,a+\omega+1,a+\omega)}=\frac{b_{12}(a+\omega+1)-b_{21}}{b_{11}(a+\omega+1)},\\ \frac{N(a+\omega+1)}{L(a+\omega+1)}&=\frac{(y_{\bar 1},y_{\bar 2};\,a+\omega+2,a+\omega+1)}{(y_{\bar 1},y_{\bar 2};\,a+\omega+1,a+\omega)}=-\frac{b_{22}}{b_{11}(a+\omega+1)}.\end{aligned}$$

Now from the difference equation $B\cdot(y_{\bar{1}},\ y_{\bar{2}};\ a+\omega+1,\ a+\omega)$ equals

$$(v_{\bar{1}},\ v_{\bar{2}};\ a+\omega+1,\ a+\omega). \tag{7}$$

But from (5), (7) equals (6) which is different from zero. Hence $(y_{\bar{1}},\ y_{\bar{2}};\ a+\omega+1,\ a+\omega)\neq 0$, that is, $y_{\bar{1}}(i+\omega)$ and $y_{\bar{2}}(i+\omega)$ are not linearly dependent over $a\leqq i\leqq a+\omega+2$. If $b_{11}(i)\equiv 0$ the result that $(y_{\bar{1}},\ y_{\bar{2}};\ a+\omega+1,\ a+\omega)$ equals a constant different from zero times (6) and hence that $y_{\bar{1}}(i+\omega)$ and $y_{\bar{2}}(i+\omega)$ are not linearly dependent over $a\leqq i\leqq a+\omega+1$ is immediate. Moreover, no definition of $L(a+2\omega+1)$, $M(a+2\omega+1)$ and $N(a+2\omega+1)$ proves necessary.

When $i>a+\omega$, the required equation is

$$\begin{aligned}(y_{\bar{1}},\ y_{\bar{2}};\ i,\ i-1)y(i+1)-(y_{\bar{1}},\ y_{\bar{2}};\ i+1,\ i-1)y(i)&\\ +(y_{\bar{1}},\ y_{\bar{2}};\ i+1,\ i)y(i-1)=0,&\end{aligned} \tag{$\bar{1}$}$$

which we adjoin to (1) as given. The coefficients are defined as desired so that if y_n is any solution there exists a solution $y_{\bar{n}}$ which satisfies (4). There is a possibility of $L(i)$ vanishing when $i>a+\omega+1$. If this should be the case we shall consider as solutions of (1) only linear combinations of $y_{\bar{1}}$ and $y_{\bar{2}}$ with constant coefficients.

(b) *The Differential Equation.*

Consider

$$L(x)y''+M(x)y'+N(x)y=0, \tag{1'}$$

where $L(x)$, $M(x)$ and $N(x)$ are defined and continuous when $a\leqq x\leqq a+\omega$ and accents denote differentiation, where, moreover, $\dfrac{M(x)}{L(x)}$ and $\dfrac{N(x)}{L(x)}$ are absolutely integrable over the interval $a\leqq x\leqq b$.

Let $g<a$ but $a-g$ arbitrarily small. Let $a_{11}(x)$ and $a_{12}(x)$ be absolutely integrable over $a\leqq x\leqq a+\omega$, $a_{11}(a)$ and $a_{12}(a)$ not both zero and $a'_{11}(a)$ and $a'_{12}(a)$ existent. Let $b_{11}(x)/b_{12}(x)$ be absolutely integrable when $a+\omega\leqq x\leqq a+2\omega$ and b_{11} and b_{12} both differentiable. Let, moreover,

$$\begin{aligned}u_n(x)&=a_{11}(x)y'_n(x)+a_{12}(x)y_n(x),\\ v_n(x)&=b_{11}(x)y'_n(x)+b_{12}(x)y_n(x).\end{aligned}$$

We desire that L, M and N be defined when $g\leqq x<a$ and $a+\omega\leqq x<a+2\omega$ that b_{11} and b_{12} be defined as differentiable functions when $a+\omega-g\leqq x\leqq a+\omega$ and a_{11} and a_{12} when $g\leqq x<a$ so that if y_n is any solution of (1′) there exists a solution $y_{\bar{n}}$ such that when $g\leqq x\leqq a+\omega$

$$u_n(x)=v_{\bar{n}}(x+\omega). \tag{4'}$$

Let y_1 and y_2 be linearly independent solutions of (1′) and adopt the notation $[r(x), s(x)\,;\, k, l] \equiv r^{(k)}(x)\, s^{(l)}(x) - r^{(l)}(x)\, s^{(k)}(x)$ where the superscripts denote differentiation. Regard

$$u_j(x) \equiv v_{\bar{j}}(x+\omega), \qquad j=1,2, \tag{5′}$$

as differential equations in $y_{\bar{1}}(x+\omega)$ and $y_{\bar{2}}(x+\omega)$ when $a \leqq x \leqq b$. They are clearly solvable. Then regard (5′) as linear algebraic equations in a_{11} and a_{12} when $g \leqq x \leqq a$ for any definition of b_{11} and b_{12} and of the coefficients. Since y_1 and y_2 are linearly independent, a_{11} and a_{12} are determined $g \leqq x \leqq a$ when $a-g$ is small enough. We desire, moreover, that $y_{\bar{1}}(x+\omega)$ and $y_{\bar{2}}(x+\omega)$ be linearly independent. This will be the case if $[y_{\bar{1}}(x+\omega), y_{\bar{2}}(x+\omega)\,;\, 1, 0] \neq 0$ in the neighborhood of $x=a$. To this end we define the coefficients over $g \leqq x < a$ so that when $x \rightarrow a-0$,

$$[u_1(x), u_2(x)\,;\, 1, 0] \rightarrow A[y_1(a), y_2(a)\,;\, 1, 0], \tag{8}$$

where $A \neq 0$. First supposing that $a_{11}(a) \neq 0$ this is done by choosing two numbers a_{21} and a_{22} such that $a_{11}(a)a_{22} - a_{12}(a)a_{21} = A$, and then letting L, M and N be defined in such a way that as $x \rightarrow a-0$,

$$\frac{M(x)}{L(x)} \rightarrow \frac{a'_{11}(a)+a_{12}(a)-a_{21}}{a_{11}(a)} \text{ and } \frac{N(x)}{L(x)} \rightarrow \frac{a'_{12}(a)-a_{22}}{a_{11}(a)}.$$

Substitution shows that this definition is sufficient for (8). If $a_{11}(a)=0$, no definition of the coefficients in (1′) when $x<a$ will prove necessary. We let $g=a$.

We next desire that as $x \rightarrow a+\omega+0$,

$$[v_1(x+\omega), v_2(x+\omega)\,;\, 1, 0] \rightarrow B[y_1(a+\omega), y_2(a+\omega), 1, 0], \tag{9}$$

where $B \neq 0$. Here if $b_{11} \neq 0$ due to the arbitrariness of $y_{\bar{1}}(a+\omega)$ and $y_{\bar{2}}(a+\omega)$ we can proceed in a manner exactly analogous to that employed in the case of the difference equation choosing two numbers b_{21} and b_{22} such that $b_{11}(a+\omega)\, b_{22} - b_{12}(a+\omega)\, b_{21} = B$. One can then immediately assure himself, and details are omitted, that it is possible to define $b_{11}(x)$ and $b_{12}(x)$ as differentiable functions over $a+\omega-g \leqq x \leqq a+\omega$, so that if y_n is any solution of (1′), $\frac{d}{dx} v_n(x)$ is continuous at $a+\omega$, and hence so that the limit expressed by (9) is the same if the approach is from above or below. Let b_{11} and b_{12} be so defined, the desired conclusions are then drawn as under (a). If $b_{11} \equiv 0$ (9) is immediate and obvious without special definition of $y_{\bar{1}}(a+\omega)$ and $y_{\bar{2}}(a+\omega)$ which are no longer arbitrary.

The required equation when $g < x \leqq a+2\omega$ is

$$[y_{\bar{1}}(x), y_{\bar{2}}(x)\,;\, 1, 0]\, y'' - [y_{\bar{1}}(x), y_{\bar{2}}(x)\,;\, 2, 0]\, y' + [y_{\bar{1}}(x), y_{\bar{2}}(x)\,;\, 2, 1]\, y = 0. \tag{1′}$$

If $[y_{\bar{1}}(x), y_{\bar{2}}(x)\,;\, 1, 0]$ should vanish we proceed as under (a) considering as solutions only linear combinations of $y_{\bar{1}}$ and $y_{\bar{2}}$ with constant coefficients.

§ 2. THE CHARACTERISTIC EQUATION.

I shall now make the substitution of the letter "x" for i of (a) of section 1 and x of (b). The reasoning applies to either the difference or the differential equation and the following notation with use of x is to include both. Other notation will be explained.

We propose the question: *Does a solution of* (1) [(1′)] *exist such that*

$$\rho u_n(x) = v_n(x+\omega)$$

where ρ *is a constant?*

Let y_1 and y_2, as previously, be two linearly independent solutions. As y_1 and y_2 are linear combinations of $y_{\bar{1}}$ and $y_{\bar{2}}$ we write

$$v_1(x+\omega) = \alpha_{11} u_1(x) + \alpha_{12} u_2(x),$$
$$v_2(x+\omega) = \alpha_{21} u_1(x) + \alpha_{22} u_2(x).$$

Since from the definition of the coefficients the Wronskian determinant of $v_1(x)$ and $v_2(x)$ is different from zero at $a+\omega$, $\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21} \neq 0$.

Assume now the existence of a solution $y_3(x) = \beta_1 y_1(x) + \beta_2 y_2(x) \not\equiv 0$, such that

$$v_3(x+\omega) = \rho u_3(x).$$

Then

$$v_3(x+\omega) = \beta_1 v_1(x+\omega) + \beta_2 v_2(x+\omega) = \beta_1 \{\alpha_{11} u_1(x) + \alpha_{12} u_2(x)\}$$
$$+\beta_2\{\alpha_{21} u_1(x) + \alpha_{22} u_2(x)\} = \rho u_3(x) = \rho \beta_1 u_1(x) + \rho \beta_2 u_2(x).$$

Transposing,

$$[\beta_1(\alpha_{11}-\rho) + \beta_2 \alpha_{21}] u_1(x) + [\beta_1 \alpha_{12} + \beta_2(\alpha_{22}-\rho)] u_2(x) = 0.$$

Since $y_1(x)$ and $y_2(x)$ are linearly independent, $u_1(x)$ and $u_2(x)$ are linearly independent and hence necessarily

$$\beta_1(\alpha_{11}-\rho) + \beta_2 \alpha_{21} = 0,$$
$$\beta_1 \alpha_{12} + \beta_2(\alpha_{22}-\rho) = 0.$$

But β_1 and β_2 are not both zero as $y_3(x) \not\equiv 0$, and consequently,

$$D_1 = (\alpha_{11}-\rho)(\alpha_{22}-\rho) - \alpha_{12}\alpha_{21} = 0. \tag{10}$$

This has been derived as a necessary condition. However, if ρ is so chosen that $D_1=0$ we can retrace the steps and see that a solution satisfying the relation $v_n(x) \equiv \rho u_n(x)$ does exist.

Regard (10) as an equation in ρ. We shall call it the characteristic equation of (1) [(1′)]. It is of the second degree and neither root is zero as $\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21} \neq 0$. There will always, then, exist at least one solution of (1) [(1′)], not identically zero, satisfying the relation $v_n(x+\omega) = \rho u_n(x)$.

If equation (10) has two distinct roots, ρ_1 and ρ_2, there exist at least two solutions, y_4 and y_5, not identically zero, satisfying respectively the relations

$$v_4(x+\omega) = \rho_1 u_4(x) \text{ and } v_5(x+\omega) = \rho_2 u_5(x).$$

Moreover, y_4 and y_5 are linearly independent for, assuming the contrary, there exist two constants μ_1 and μ_2 not both zero such that $\mu_1 y_4 + \mu_2 y_5 \equiv 0$; hence, such that

$$\mu_1 u_4(x) + \mu_2 u_5(x) \equiv 0 \quad \text{and} \quad \mu_1 v_4(x+\omega) + \mu_2 v_5(x+\omega) \equiv 0. \tag{11}$$

From the last

$$\mu_1 \rho_1 u_4(x) + \mu_2 \rho_2 u_5(x) \equiv 0. \tag{12}$$

For definiteness assume $\mu_1 \neq 0$ and eliminate $u_5(x)$ from (11) and (12). We get

$$\mu_1(\rho_1 - \rho_2) u_4(x) \equiv 0,$$

but $u_4(x) \not\equiv 0$; for it is surely possible to choose a solution y_6 linearly independent of y_4, and hence, according to section 1, such that $u_6(x)$ is linearly independent of $u_4(x)$. But this is impossible if $u_4(x) \equiv 0$. It results that $\rho_1 = \rho_2$, a contradiction.

§ 3. INDEPENDENCE OF THE CHARACTERISTIC EQUATION OF THE PARTICULAR FUNDAMENTAL SYSTEM OF SOLUTIONS CHOSEN.

Consider a second fundamental system of solutions y_7 and y_8. Then

$$v_7(x+\omega) = B_{11} u_7(x) + B_{12} u_8(x),$$
$$v_8(x+\omega) = B_{21} u_7(x) + B_{22} u_8(x).$$

The characteristic equation is

$$D_2 = (B_{11} - \rho)(B_{22} - \rho) - B_{12} B_{21} = 0.$$

But

$$y_7(x) = L_{11} y_1(x) + L_{12} y_2(x),$$
$$y_8(x) = L_{21} y_1(x) + L_{22} y_2(x),$$

where $L_{11} L_{22} - L_{12} L_{21} \neq 0$. Hence,

$$v_7(x+\omega) = B_{11}[L_{11} u_1(x) + L_{12} u_2(x)] + B_{12}[L_{21} u_1(x) + L_{22} u_2(x)].$$

On the other hand, one can write

$$\begin{aligned} v_7(x+\omega) &= L_{11} v_1(x+\omega) + L_{12} v_2(x+\omega) \\ &= L_{11}[\alpha_{11} u_1(x) + \alpha_{12} u_2(x)] + L_{12}[\alpha_{21} u_1(x) + \alpha_{22} u_2(x)]. \end{aligned}$$

Equate and collect:

$$\begin{aligned} [(B_{11} L_{11} + B_{12} L_{21}) - (L_{11} \alpha_{11} + L_{12} \alpha_{12})] u_1(x) + [(B_{11} L_{12} + B_{12} L_{22}) \\ - (L_{11} \alpha_{12} + L_{12} \alpha_{22})] u_2(x) = 0. \end{aligned}$$

Here the coefficients must be zero as $u_1(x)$ and $u_2(x)$ are linearly independent; that is,

$$B_{11} L_{11} + B_{12} L_{21} = L_{11} \alpha_{11} + L_{12} \alpha_{21},$$
$$B_{11} L_{12} + B_{12} L_{21} = L_{11} \alpha_{12} + L_{12} \alpha_{22}.$$

Similarly,

$$B_{21} L_{11} + B_{22} L_{21} = L_{21} \alpha_{11} + L_{22} \alpha_{21},$$
$$B_{21} L_{12} + B_{22} L_{22} = L_{21} \alpha_{12} + L_{22} \alpha_{22}.$$

Hence, $D_1 = D_2$.* That is: *The characteristic equation is independent of the particular fundamental system of solutions chosen.*

§ 4. THE ROOTS OF THE CHARACTERISTIC EQUATION COINCIDENT.

Next suppose that the roots of the characteristic equation coincide. Denote this common value by ρ_1. We know that there exists at least one solution $y_4(x)$ not identically zero, such that $v_4(x+\omega) \equiv \rho_1 u_4(x)$. Let y_9 be a particular solution linearly independent of y_4, then

$$v_4(x+\omega) = \rho_1 u_4(x),$$
$$v_9(x+\omega) = c_{21} u_4(x) + c_{22} u_9(x).$$

The characteristic equation is

$$(\rho_1 - \rho)(c_{22} - \rho) = 0.$$

This must have ρ_1 a double root; hence $c_{22} = \rho_1$, that is,

$$v_9(x+\omega) = c_{21} u_4(x) + \rho_1 u_9(x), \tag{13}$$

a relation that will subsequently prove of importance.

§ 5. REAL AND IMAGINARY SOLUTIONS WHEN L, M AND N ARE REAL.

If ρ_1 and ρ_2 are distinct and y_4 and y_5 are as in section 2, one can establish the following theorems. The proofs being extremely easy are omitted.

If ρ_1 *and* ρ_2 *are real, then* y_4 *and* y_5 *are real but for possible constant multipliers.*

When ρ_1 *and* ρ_2 *are imaginary,* y_4 *and* y_5 *are conjugate but for a possible constant multiplier.*

When ρ_1 *and* ρ_2 *are imaginary and* y_4 *and* y_5 *taken conjugate, in order for a solution* $C_1 y_4 + C_2 y_5$ *to be real, it is necessary and sufficient that* C_1 *and* C_2 *be conjugate.*

Similarly, we can easily show that: *If* ρ_1 *and* ρ_2 *are equal they are real, and a solution satisfying the relation* $v_4(x+\omega) = \rho_1 u_4(x)$ *is real but for a possible constant multiplier.*

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* "Theory of Differential Equations," Forsyth, Vol. IV, Part III, p. 39.

PART II.

§ 6. FUNDAMENTAL BOUNDARY-VALUE THEOREM FOR THE DIFFERENCE EQUATION.

Consider

$$\Delta\{K(i,\lambda)\,\Delta y(i)\}-G(i,\lambda)\,y(i+1)=0,^{*} \tag{14}$$

where $K(i,\lambda)>0$ is defined when $a\leqq i\leqq a+\omega$ and $G(i,\lambda)$ when $a\leqq i\leqq a+\omega-1$ for all values of the real parameter λ. Where, moreover, G is continuous in λ and such that when λ increases from $-\infty$ to ∞, it continuously increases going from $-\infty$ to ∞ also and where K is continuous in λ, never decreases and approaches a limit when $\lambda\rightarrow\infty$.

In writing G and K the argument λ will frequently be omitted. Denote an increment of λ by $\delta\lambda$ and let $\bar{\lambda}=\lambda+\delta\lambda$. Denote the corresponding functions by a (—) thus $\bar{G}$, $\bar{y}$, etc. One can now establish from the difference equation the following well-known fundamental relation. Where k is any integer $a<k\leqq a+\omega$

$$\begin{aligned}\bar{K}(k)\Delta\bar{y}(k)y(k)-K(k)\Delta y(k)\bar{y}(k)-[\bar{K}(a)\Delta\bar{y}(a)y(a)-K(a)\Delta y(a)\bar{y}(a)]\\ =\sum_{i=a}^{k-1}(\bar{K}(i)-K(i))\Delta\bar{y}(i)\Delta y(i)+\sum_{i=a}^{k-1}(\bar{G}(i)-G(i))\bar{y}(i+1)y(i+1).\end{aligned} \tag{15}$$

From this one concludes that if $y(a)$ and $\Delta y(a)$ are continuous in λ and are only allowed to vary so that

$$\bar{K}(a)\Delta\bar{y}(a)y(a)-K(a)\Delta y(a)\bar{y}(a)\geqq 0. \tag{16}$$

that if $\delta\lambda>0$ is sufficiently small $\bar{K}(k)\Delta\bar{y}(k)y(k)-K(k)\Delta y(k)\bar{y}(k)>0$, that is, that $\dfrac{K(k)\Delta y(k)}{y(k)}$ increases with λ when defined and that since when $y(k)=0$ and $|\delta\lambda|>0$ sufficiently small $-K(k)\Delta y(k)\bar{y}(k)\neq 0$ roots of $y(k)$ do not cluster. One can now conclude without difficulty that nodes† of y move continuously to the right as λ increases.

We now define y and K as continuous functions; namely, those functions defined by the broken line graphs of $y(i)$ and $K(i)$, respectively. Let y' denote the forward derivative of y. Let, moreover, $\bar{x}$ be a fixed point, $a\leqq\bar{x}<a+1$. When if $c_1y'(\bar{x})+c_2y(\bar{x})\equiv 0$ where c_1 and c_2 are independent of λ, the assumption that y does not satisfy (16) leads to an immediate contradiction. Hence, if when λ varies, we require that $c_1y'(\bar{x})+c_2y(\bar{x})=0$, we thereby require that (16) be satisfied.

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* An equation $L(i)\,y(i+1)+M(i)\,y(i)+N(i)\,y(i-1)=0$ where $L(i)\,M(i)>0$, can be written in the form $\Delta\{K(i)\,\Delta y(i)\}-G(i)\,y(i+1)=0$ where $K(i)>0$; and, conversely.

† A real function defined only at integral points can be plotted in the ordinary Cartesian plane as a succession of isolated points. Join these points with straight line segments. The whole will, in general, form a broken line which will be spoken of as the broken line graph of the function. The points where it crosses the axis are called nodes.

One can now conclude in various ways* that when $\lambda > 0$ is sufficiently large a solution satisfying $c_1 y'(\bar{x}) + c_2 y(\bar{x}) = 0$ has no node on $a+1 < x \leqq a+\omega+1$ and that $\dfrac{\Delta y(a+\omega)}{y(a+\omega)}$ is positive and as large as we wish to make it, and that, moreover, when $\lambda < 0$ is numerically sufficiently large not only are there the maximum number of nodes on $a+1 < x \leqq a+\omega+1$ but that the last is as close to $a+\omega$ as we desire, that is, $\dfrac{\Delta y(a+\omega)}{y(a+\omega)}$ is negative and numerically as large as we wish to have it.

Moreover, the requirement that $y(\bar{x}+\omega) = 0$ is equivalent to a requirement of the form $\bar{c}_1 K(a+\omega)\,\Delta y(a+\omega) + \bar{c}_2\, y(a+\omega) = 0$. We, consequently, conclude:

Let y *be a solution satisfying* $c_1 y'(\bar{x}) + c_2 y(\bar{x}) = 0$. *There exist values of* λ, $\lambda_0 > \lambda_1 > \ldots . > \lambda_k$ *such that when* $\lambda = \lambda_j$, $j = 0, \ldots ., k$, $y(\bar{x}+\omega) = 0$ *also; where, moreover,* y *has exactly* j *nodes on the interval* $a+1 < x \leqq a+\omega$. *If* $y(a+1) \neq 0$, $k = \omega - 1$, *if* $y(a+1) = 0$, $k = \omega - 2$.

§ 7. EXTENSION OF FUNDAMENTAL BOUNDARY-VALUE THEOREM TO EQUATIONS OF THE TYPE SET UP IN SECTION 1.

(a) *The Difference Equation.*

Write equation (14) in the form

$$L(i,\lambda)\, y(i+1) + M(i,\lambda)\, y(i) + N(i,\lambda)\, y(i-1) = 0. \tag{17}$$

Here $L(i,\lambda)$, $M(i,\lambda)$ are defined for all values of λ when $a+1 \leqq i \leqq a+\omega$. We let $u_n(i) = y_n(i-1)$, $v_n(i+\omega) = d_{11}(i)\, K(a+\omega)\, \Delta y_n(i+\omega) + d_{12}(i)\, y_n(i+\omega)$, where $d_{11}(i)$ and $d_{12}(i)$ are real functions defined when $a \leqq i \leqq a+\omega+1$, $d_{11}(i) \neq 0$ or $d_{11} \equiv 0$. We then define L, M and N according to the method of section 1 so that if y_n is any solution there exists a solution $y_{\bar{n}}$ such that $u_n(i) = v_{\bar{n}}(i+\omega)$. u_n *and* $v_{\bar{n}}$ *are to be considered functions of the continuous variable* x, *namely, those functions defined by the broken line graphs of* $u_n(i)$ *and* $v_n(i)$, *respectively.* We write moreover in place of the $a_{21} y_n(a+1) + a_{22} y_n(a)$ of section 1 $K(a)\,\Delta y_n(a) + b\, y_n(a) = \bar{u}_n(a+1)$ and in place of the $b_{21} y_n(a+\omega+1) + b_{22} y_n(a+\omega)$ of section 1 $d_{21} K(a+\omega)\,\Delta y_n(a+\omega) + d_{22} y_n(a+\omega) = \bar{v}_n(a+\omega)$ where $d_{11} d_{22} - d_{12} d_{21} = 1$.

Let $a+1 \leqq \bar{x} \leqq a+\omega+1$ and let y_n denote a particular solution, not identically zero, such that $u_n(\bar{x}) = 0$. Denote by c that integer such that $c < \bar{x} \leqq c+1$. Form the equation

$$\bar{L}(i,\lambda)\, y(i+1) + \bar{M}(i,\lambda)\, y(i) + \bar{N}(i,\lambda)\, y(i-1) = 0, \tag{18}$$

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* A satisfactory method of proving this theorem is by comparison with $\zeta\Delta^2 y(i) - P(i)\, y(i+1) = 0$ where P is a constant chosen large numerically and positive or negative as we desire, and ζ is a constant such that $0 < \zeta < K(i)$.

where $\bar{L}(i,\lambda)$, $\bar{M}(i,\lambda)$ and $\bar{N}(i,\lambda)$ are equal respectively to $L(i,\lambda)$, $M(i,\lambda)$ and $N(i,\lambda)$ when $c+1\leqq i\leqq a+\omega$ and to $L(i-\omega-1,\lambda)$, $M(i-\omega-1,\lambda)$ and $N(i-\omega-1,\lambda)$ respectively when $a+\omega+1\leqq i\leqq c+\omega+1$.

Now denote by y_{10} that solution of (14) such that $v_n(i+\omega)=u_{10}(i)$. Then there exists a solution of (18) which we denote by y_{11} such that $y_{11}(i)=y_{10}(i-\omega-1)$ over $a+\omega\leqq i\leqq c+\omega+2$. Therefore, $u_{11}(a+\omega+2)=v_n(a+\omega+1)$ and $u_{11}(a+\omega+1)=v_n(a+\omega)$. Due to the definition given $L(a,\lambda)$, $M(a,\lambda)$, $N(a,\lambda)$, $L(a+\omega+1,\lambda)$, $M(a+\omega+1,\lambda)$ and $N(a+\omega+1,\lambda)$, these equations reduce to

$$\left.\begin{aligned}\bar{u}_{11}(a+\omega+2)&=v_n(a+\omega),\\ u_{11}(a+\omega+2)&=\bar{v}_n(a+\omega).\end{aligned}\right\}\qquad(20)$$

Hence since y_n satisfies a relation like (16) at $a+\omega$, y_{11} satisfies the same relation at $a+\omega+1$.

Solving (20) $y_{11}(a+\omega+2)=\bar{D}K(a+\omega)\Delta y_n(a+\omega)+Dy_n(a+\omega)$. We shall say that we have case α if $\bar{D}\neq 0$ and case β if $\bar{D}=0$. Apply the theorem of oscillation for y_n and we have that as λ increases from $-\infty$ to ∞ $y_{11}(a+\omega+2)$ vanishes exactly m times where

CASE α: (a) $y_n(c+1)\neq 0$, $m=a+\omega-c$,
(b) $y_n(c+1)=0$, $m=a+\omega-c-1$;
CASE β: (a) $y_n(c+1)\neq 0$, $m=a+\omega-c-1$,
(b) $y_n(c+1)=0$, $m=a+\omega-c-2$.

I shall next show that: *If λ is positive and sufficiently large y_{11} has no node on $a+\omega+2<x\leqq\bar{x}+\omega$ and if λ is negative and numerically sufficiently large, the maximum number namely, $c-a-2$.*

CASE α. Here $\dfrac{\Delta y_{11}(a+\omega+1)}{y_{11}(a+\omega+2)}$ approaches a limit as λ becomes infinite and one can immediately prove the theorem, as for example, as indicated in the third footnote under section 6.

CASE β. First, when λ is positive and sufficiently large a solution of (18) exists having no node* on $a+\omega+1\leqq x\leqq c+\omega+1$. Hence as the nodes of two solutions either separate each other or coincide, no solution can have more than one node on this interval. But under these circumstances, as $y_{11}(a+\omega+2)$ is proportional to $y_n(a+\omega)$, $y_{11}(a+\omega+1)$ which is also a linear combination of $K(a+\omega)\Delta y_n(a+\omega)$ and $y_n(a+\omega)$ can not be proportional to $y_n(a+\omega)$ also consistent with the equation $d_{11}(a)d_{22}-d_{21}d_{12}(a)=1$; and hence $y_{11}(a+\omega+1)$ vanishes one more time than $y_{11}(a+\omega+2)$ as λ increases from $-\infty$ to ∞, and as when $x \geqq a+\omega+1$ its nodes move to the left as λ decreases and to the right as λ increases when $\lambda=\infty$ there must remain one node of y_{11} on the interval $a+\omega+1<x<a+\omega+2$, and hence when λ is sufficiently large y_{11} can have no node on $a+\omega+2<x \leqq c+\omega+1$.

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* This will be the true of any solution having the same sign at $a+\omega+1$ and $a+\omega+2$.

Next, when λ is negative and numerically sufficiently large a solution exists having the maximum number of nodes* on $a+\omega+1 \leqq x \leqq c+\omega+1$ conceivably possible, and hence y_{11} surely has on this interval at least as many nodes as one less than this maximum number. But as λ decreases from ∞ to $-\infty$ $y_{11}(a+\omega+1)$ vanishes one more time than $y_{11}(a+\omega+2)$, and hence when λ is negative and numerically sufficiently large y_{11} has no node on the interval $a+\omega+1 \leqq x \leqq a+\omega+2$ and hence the maximum number of nodes on $a+\omega+2 < x \leqq c+\omega+1$.

Now bearing in mind that as λ decreases nodes of y_{11} on the interval $a+\omega+1 \leqq x \leqq c+\omega+1$ move continuously to the left, that

$$\frac{K(a+\omega)\,\Delta y_{11}(a+\omega)}{y_{11}(a+\omega)} \rightarrow -\infty$$

as $\lambda \rightarrow -\infty$ and that $y_{11}(\bar{x}+\omega) = u_{11}(\bar{x}+\omega+1) = u_{11}(\bar{x}) = v_n(\bar{x}+\omega)$ a mere count gives the following theorem:

There exist values of λ, $\lambda_0>\lambda_1>\ldots.>\lambda_k$ *such that when* $\lambda=\lambda_j$, $j=0, \ldots., k$, *each solution of* (14) *satisfying the relation* $y_n(\bar{x}-1)=0$ *satisfies also the relation* $v_n(\bar{x}+\omega)=0$.

CASE α: (a) $k=\omega-1$,
(b) $k=\omega-2$;
CASE β: (a) $k=\omega-2$,
(b) $k=\omega-3$.

The notation "λ_j*" for any particular one of the values* $\lambda_0, \ldots., \lambda_k$ *will be generally used.*

(b) *The Differential Equation.*

We consider the equation

$$\frac{d}{dx}(K(x,\lambda)\,y'(x)) - G(x,\lambda)\,y(x) = 0,$$

where the prime denotes differentiation and where K and G are defined and continuous for all values of λ when $a \leqq x \leqq a+\omega$ and where moreover $G(x,\lambda)$ is continuous in λ and continually increases going from $-\infty$ to ∞ as λ increases from $-\infty$ to ∞ and where $K>0$ never decreases.

---

* This will be the true of any solution having opposite signs at $a+\omega+1$ and $a+\omega+2$.

We write this equation in the form

$$L(x, \lambda)y'' + M(x, \lambda)y' + N(x, \lambda)y = 0$$

and define L, M and N over the interval $a+\omega < x \leqq a+2\omega$, according to the method of section 1, so that if y_n is any solution there exists a solution $y_{\bar n}$ such that $u_n(x) \equiv v_{\bar n}(x+\omega)$ where

$$u_n(x) = y_n(x) \text{ and } v_{\bar n}(x+\omega) = d_{11}(x)K(a+\omega)y'_{\bar n}(x+\omega) + d_{12}(x)y_{\bar n}(x+\omega)$$

where d_{11} and d_{12} are given functions as in section 1 (b). The analogue of (a) of this section is then immediate. Form the equation

$$\bar L(x, \lambda)y'' + \bar M(x, \lambda)y' + \bar N(x, \lambda)y = 0, \tag{18'}$$

where $\bar L = L$, $\bar M = M$ and $\bar N = N$ when $\bar x \leqq x \leqq a+\omega$ and to $L(x-\omega, \lambda)$ etc., when $a+\omega \leqq x \leqq \bar x+\omega$ and reason as under (a).

There exist an infinite number of real values of λ, $\lambda_0 > \lambda_1 > \lambda_2 > \ldots$, *such that when* $\lambda = \lambda_j$, $j = 0, 1, 2, \ldots$, *all solutions satisfying* $u_n(\bar x) = 0$ *satisfy also* $v_n(\bar x+\omega) = 0$; $\bar x$ *any particular point* $a \leqq x \leqq a+\omega$.

§ 8. CONTINUITY PROOF.

(a) *The Difference Equation.*

It is apparent that the λ_j's of section 7 will, in general, be functions of the point $\bar x$. We write $\lambda_j(\bar x)$ and *shall prove that these functions, with the exception of the last* λ_k, *are continuous* $a+1 \leqq \bar x \leqq a+\omega+1$ *and that* λ_k *is continuous over every interval,* $i < \bar x < i+1$, *where* i *is an integer becoming negatively infinite as* $\bar x$ *approaches* i *or* $i+1$.

Let ξ be any particular fixed value of $\bar x$ and let y_ξ be a particular solution, not identically zero, such that $u_\xi(\xi) = y_\xi(\xi-1) = 0$. Let $\delta > 0$ and let $0 < \xi - \bar x < \delta$; and then let $y_{\bar x}$ be a solution satisfying the conditions $y_{\bar x}(\bar x - 1) = 0$ and $y'_{\bar x}(\bar x - 1) = y'_\xi(\xi - 1)$ where as previously the accent denotes the forward derivative. Let c be that integer such that $c < \xi - 1 \leqq c+1$ and let $\varepsilon > 0$ be arbitrarily small. Then, if δ is small enough $|y_{\bar x}(c) - y_\xi(c)| < \varepsilon$ and $|y_{\bar x}(c+1) - y_\xi(c+1)| < \varepsilon$. But $y_\xi(a+\omega)$ and $y_\xi(a+\omega+1)$ are polynomials in $y_\xi(c)$ and $y_\xi(c+1)$; and moreover, $y_{\bar x}(a+\omega)$ and $y_{\bar x}(a+\omega+1)$ are the same polynomials in $y_{\bar x}(c)$ and $y_{\bar x}(c+1)$. As polynomials are continuous, if ε be sufficiently small $y_{\bar x}(a+\omega)$ and $y_{\bar x}(a+\omega+1)$ can be made to differ in absolute value as little as we please from $y_\xi(a+\omega)$ and $y_\xi(a+\omega+1)$, respectively.

Now consider equation (18) formed for c and the solutions of it $\bar{\bar y}_\xi$ and $\bar{\bar y}_{\bar x}$, bearing the relations to y_ξ and $y_{\bar x}$, respectively, expressed by (20) and hence where $\eta > 0$ is arbitrary, if ε is small enough $|\bar{\bar y}_\xi(a+\omega+1) - \bar{\bar y}_{\bar x}(a+\omega+1)| < \eta$

and $|\bar{\bar{y}}_{\xi}(a+\omega+2)-\bar{\bar{y}}_{\bar{x}}(a+\omega+2)|<\eta$. Moreover, for larger values of i $\bar{\bar{y}}_{\xi}$ and $\bar{\bar{y}}_{x}$ are polynomials in these values.

Let $\lambda=\lambda_j(\xi)$, $j<k$ if $\xi=c+1$, otherwise $j\leqq k$; then a node which we call the $(j+1)$-st node of $\bar{\bar{y}}_{\xi}$ lies at $\xi+\omega$. Let $\theta>0$, then if η be small enough, the $(j+1)$-st node of $\bar{\bar{y}}_{x}$ lies a distance from $\bar{x}+\omega$ less than θ. Now hold $\bar{x}$ fast and vary λ. As nodes of $\bar{\bar{y}}_{\bar{x}}$ move continuously to the right as λ increases, and to the left as λ decreases; by a variation in λ as small as we please, θ sufficiently small, the $(j+1)$-st node of $\bar{\bar{y}}_{\bar{x}}$ can be made to move to $\bar{x}+\omega$. When it will have moved to this point, λ will have the value $\lambda_j(\bar{x})$. That is given a $\zeta>0$ we can choose a θ, then an η, then an ε, then a δ so that $|\lambda_j(\bar{x})-\lambda_j(\xi)|<\zeta$ when $0<\xi-\bar{x}<\delta$.

Now let $0\leqq\bar{x}-\xi<\bar{\delta}$ and let c be that integer such that $c\leqq\xi-1<c+1$. We form equation (18) for this point c. The corresponding theory developed with the relations between the y's and $\bar{y}$'s now holds without the slightest change. In fact, it was quite immaterial whether we chose c that integer $c<\bar{x}-1\leqq c+1$, as we did or $c\leqq\bar{x}-1<c+1$. The repetition of the reasoning just gone through now shows that given a $\zeta>0$ it is possible to choose a $\bar{\delta}>0$ so that when $\bar{x}-\xi<\bar{\delta}$, $|\lambda_j(\bar{x})-\lambda_j(\xi)|<\zeta$.

Let $\bar{\bar{\delta}}\leqq\delta$, $\bar{\delta}$ then given a $\zeta>0$ it is possible to choose a $\bar{\bar{\delta}}>0$ such that when $|\bar{x}-\xi|<\bar{\bar{\delta}}$, $|\lambda_j(\bar{x})-\lambda_j(\xi)|<\zeta$. This establishes continuity.

The exceptional situation in the case of λ_k arises from the fact that if $\bar{x}-1$ is integral, there exists no value λ_k. We have case α (b) or case β (b). Let $\bar{x}-1$ approach an integer $c+1$ from below. Consider (18) formed for c. The value of λ which makes $\bar{y}(a+\omega+2)$ vanish the $(a+\omega-c)$-th or the $(a+\omega-c-1)$-st time, as we have case α (b) or case β (b), becomes negatively infinite; but at all times λ is less than or equal to this value and hence itself becomes negatively infinite. Similarly, if $\bar{x}-1$ approaches an integer c from above we think of (18) where c is that integer $c\leqq\bar{x}-1<c+1$ with exactly the same results as when $\bar{x}-1$ approaches $c+1$ from below.

(b) *The Differential Equation.*

All the values of $\lambda_j(\bar{x})$ *are continuous* $a\leqq\bar{x}\leqq a+\omega$ *in the case of the differential equation.*

Here we make use of a few fundamental and well-known theorems for the differential equation and follow the general method employed in (a). There is less need for detail; and naturally no supplementary discussion for a particular λ_j is necessary as for the λ_k under (a). I shall again for the sake of brevity omit details of the proof.

PART III.

§ 9. STATEMENT OF PROBLEM.

(a) *The Difference Equation.*

Consider again

$$\Delta\{K(i,\lambda)\,\Delta y(i)\} - G(i,\lambda)\,y(i+1) = 0, \tag{21}$$

where $K(i,\lambda) > 0$ and $G(i,\lambda)$ are defined when

$$a \leqq i \leqq a+\omega \tag{22}$$

and

$$a \leqq i \leqq a+\omega-1 \tag{23}$$

respectively, where moreover K and G are continuous in the real parameter λ, when $-\infty < \lambda < \infty$ and such that as λ increases G always actually increases going from $-\infty$ to ∞ and K does not decrease approaching a limit as $\lambda \rightarrow \infty$.

We shall consider (21) subject to the conditions

$$\left.\begin{aligned} \alpha_{11} K(a)\,\Delta y(a) + \alpha_{12} y(a) &= \beta_{11} K(a+\omega)\,\Delta y(a+\omega) + \beta_{12}\, y(a+\omega), \\ \alpha_{21} K(a)\,\Delta y(a) + \alpha_{22} y(a) &= \beta_{21} K(a+\omega)\,\Delta y(a+\omega) + \beta_{22}\, y(a+\omega), \end{aligned}\right\} \tag{24}$$

where $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = \beta_{11}\beta_{22} - \beta_{21}\beta_{12}$.

It proves advisable to throw conditions (24) into the form

$$\left.\begin{aligned} K(a)\,\Delta y(a) + b\, y(a) &= d_{11} K(a+\omega)\Delta y(a+\omega) + d_{12}\, y(a+\omega), \\ y(a) &= d_{21} K(a+\omega)\Delta y(a+\omega) + d_{22}\, y(a+\omega), \end{aligned}\right\} \tag{25}$$

where $d_{11} \neq 0$, $d_{11} d_{22} - d_{21} d_{12} = 1$.

It will prove advantageous also to consider side by side with (25)

$$\left.\begin{aligned} K(a)\,\Delta y(a) + b y(a) &= -d_{11} K(a+\omega)\,\Delta y(a+\omega) - d_{12}\, y(a+\omega), \\ y(a) &= -d_{21} K(a+\omega)\,\Delta y(a+\omega) - d_{22} y(a+\omega). \end{aligned}\right\} \tag{26}$$

Let $u_n(i) = y_n(i-1)$ and $v_n(i+\omega) = d_{11} K(a+\omega)\,\Delta y_n(i+\omega) + d_{12}\, y_n(i+\omega)$. We then define the coefficients as in section 1 so that if y_n is any solution of (21) there exists a solution $y_{\bar{n}}$ such that $u_n(i) \equiv v_{\bar{n}}(i+\omega)$ and also $u_n(a) = K(a)\,\Delta y_n(a) + b\, y_n(a) = \bar{u}_n(a+1)$ and $v_n(a+\omega+1) = d_{21} K(a+\omega)\,\Delta y_n(a+\omega) + d_{22}\, y_n(a+\omega) = \bar{v}_n(a+\omega)$. It is only necessary to replace the a_{21}, a_{22}, b_{21} and b_{22} of section 1 by $K(a)$, $-K(a)+b$, $d_{21}K(a+\omega)$ and $-d_{21}K(a+\omega)+d_{22}$, respectively, and proceed as there.

The existence of a solution of (21) *not identically zero satisfying* (25) *now becomes necessary and sufficient for the existence of a solution not identically zero satisfying identically the relation*

$$u_n(i) = v_n(i+\omega),$$

and the existence of a solution not identically zero satisfying (26) *becomes necessary and sufficient for the existence of a solution not identically zero satisfying identically the relation*

$$u_n(i) = -v_n(i+\omega).$$

The sufficiency of these conditions is immediate. The necessity is proven thus. Suppose that there exists a solution satisfying (25). Denote this by y_1. Then

$$\begin{aligned} K(a)\,\Delta y_1(a)+b\,y_1(a)&=d_{11}K(a+\omega)\,\Delta y_1(a+\omega)+d_{12}y_1(a+\omega)\\ y_1(a)&=d_{21}K(a+\omega)\,\Delta y_1(a+\omega)+d_{22}y_1(a+\omega). \end{aligned}$$

We know that there exists a solution $y_{\bar{1}}$ such that $u_1(i)=v_{\bar{1}}(i+\omega)$. Then

$$\begin{aligned} K(a)\,\Delta y_1(a)+b\,y_1(a)&=d_{11}K(a+\omega)\,\Delta y_{\bar{1}}(a+\omega)+d_{12}y_{\bar{1}}(a+\omega)\\ y_1(a)&=d_{21}K(a+\omega)\,\Delta y_{\bar{1}}(a+\omega)+d_{22}y_{\bar{1}}(a+\omega), \end{aligned}$$

and consequently, $y_{\bar{1}}$ is identical with y_1 at $a+\omega$ and $a+\omega+1$ and hence at all points.

We now define $u_n(i)$ and $v_n(i)$ as functions of the continuous variable x, as has previously been done; namely, as those functions defined by the broken line graphs of $u_n(i)$ and $v_n(i)$, respectively. It results that the satisfaction of $u_n(x)\equiv v_n(x+\omega)$ by a solution of (21), not identically zero, is necessary and sufficient for the satisfaction of (25) by such a solution, and the satisfaction of $u_n(x)\equiv -v_n(x+\omega)$ is necessary and sufficient for the satisfaction of (26).

If values of λ exist such that (21) and (25) are simultaneously satisfied by a function of i, not identically zero, we shall speak of them as values l; and if values exist so that (21) and (26) are simultaneously satisfied by a function of i, not identically zero as values of l'. The solutions of (21) not identically zero satisfying (25) or (26) as the case may be will be called corresponding solutions.

(b) *The Differential Equation.*

We consider here again

$$\frac{d}{dx}\{K(x,\lambda)\,y'(x)\}-G(x,\lambda)\,y(x)=0,$$

where $K(x,\lambda)>0$ and $G(x,\lambda)$ for all values of the real parameter λ are defined and continuous when

$$a\leqq x\leqq a+\omega, \tag{22'}$$

and where K and G are continuous in λ, and such that as λ increases G increases, constantly going from $-\infty$ to ∞ and K does not ever decrease.

We consider (21′) subject to

$$\left.\begin{aligned} K(a)\,y'(a)&=d_{11}K(a+\omega)\,y'(a+\omega)+d_{12}y(a+\omega)\\ y(a)&=d_{21}K(a+\omega)\,y'(a+\omega)+d_{22}y(a+\omega), \end{aligned}\right\} \tag{25'}$$

where $d_{11}d_{22}-d_{12}d_{21}=1$ and also side by side with these subject to

$$\left.\begin{aligned} K(a)\,y'(a)&=-d_{11}K(a+\omega)\,y'(a+\omega)-d_{12}y(a+\omega)\\ y(a)&=-d_{21}K(a+\omega)\,y'(a+\omega)-d_{22}y(a+\omega). \end{aligned}\right\} \tag{26'}$$

Suppose $d_{21} \neq 0$.[*] Let $u_n(x) = y_n(x)$, $v_n(x+\omega) = d_{21} K(a+\omega) \Delta y_n(i+\omega) + d_{22} y(i+\omega)$.

We proceed in a manner similar to that employed under (a) of this section, defining the coefficients of the differential equation over $g \leqq x < a$ and $a+\omega < x \leqq a+2\omega$, etc., according to section 1 so that the identical satisfaction of the relation $u_n(x) = v_n(x+\omega)$ by a solution of (21′) not identically zero is necessary and sufficient for the satisfaction of (25′) by the same solution. The identical satisfaction of $u_n(x) = -v_n(x+\omega)$ is then also necessary and sufficient for the satisfaction of (26′) by the same solution. We let

$$\bar{u}_n(a) = y'_n(a) \text{ and } \bar{v}_n(a+\omega) = \frac{1}{K(a)} [d_{11} K(a+\omega) y'(a+\omega) + d_{12} y(a+\omega)].$$

§ 10. TO PROVE THAT THE MAXIMA[†] AND MINIMA OF THE VALUES λ_j ARE VALUES l OR l'.

Under the term maximum (minimum) we include the case that the function is a constant or has a constant value over a neighborhood.

The constant term of the characteristic equation is unity since A and B of section 1 are equal.

Suppose $\lambda_j(\bar{a})$ *a maximum. To prove that when* $\lambda = \lambda_j(\bar{a})$ $\rho = \pm 1$.

Assume that this were not the case then $\rho_1 \neq \rho_2$ as $\rho_1 \rho_2 = 1$.

Denote by y_1 and y_2 two solutions, neither identically zero, such that $\rho_1 u_1(x) = v_1(x+\omega)$ and $\rho_2 u_2(x) = v_2(x+\omega)$. Then

$$u_1(x) v_2(x+\omega) - u_2(x) v_1(x+\omega) = (\rho_1 - \rho_2) u_1(x) \cdot u_2(x).$$

Let $y_n(x)$ be a solution such that $u_n(\bar{a}) = v_n(\bar{a}+\omega) = 0$, $y_n \not\equiv 0$; in the case of the difference equation $a+1 \leqq \bar{a}$ and in the case of the differential equation $a \leqq \bar{a}$. Suppose $y_n = c_1 y_1 + c_2 y_2$, then

$$0 = c_1 u_1(a) + c_2 u_2(a),$$
$$0 = c_1 \rho_1 u_1(a) + c_2 \rho_2 u_2(a).$$

As c_1 and c_2 are not both zero $(\rho_1 - \rho_2) u_1(\bar{a}) \cdot u_2(\bar{a}) = 0$. But $\rho_1 \neq \rho_2$ and hence $u_1(\bar{a}) \cdot u_2(\bar{a}) = 0$. They are not both zero and excluding $\bar{a}$ itself in some neighborhood of $\bar{a}$ neither is zero.[‡] Suppose for definiteness $u_1(\bar{a}) = 0$; then y_1 is essentially real, for write $y_1 \equiv y_j + y_k\sqrt{-1}$, where y_j and y_k are real. Then $u_j(\bar{a}) = u_k(\bar{a}) = 0$ and hence y_j and y_k are proportional. Consequently, $y_1 \equiv c y_j$. Let $y_1 \equiv y_j$. It results that ρ_1 is real, whence ρ_2 is also. Hence y_2 is essentially real and will be considered real. $u_1(x)$ changes sign at $\bar{a}$, and hence the determinant (27) changes sign at $\bar{a}$ also.

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[*] In case $d_{21} = 0$ we simply let $v_n(x+\omega) = D_{21}(x) y'_n(x+\omega) + D_{22}(x) y_n(x+\omega)$, where $D_{21}(a+\omega) = 0$, $D_{22}(a+\omega) = d_{22}$, $D'_{21}(a+\omega) + D_{22}(a+\omega) = \frac{d_{11}}{K(a)}$ and $D'_{22}(a+\omega) = \frac{d_{12}}{K(a)}$. Then let $y_{\bar{1}}$ and $y_{\bar{2}}$ be so chosen, first and second derivatives existent, that, if $D_{21}(x) y'_{\bar{1}}(x+\omega) + D_{22}(x) \bar{y}_1 = y_1$, $D_{22}(x) y'_{\bar{2}}(x-\omega) + D_{22}(x) y_{\bar{2}} = y_2$, D_{21} and D_{22} together with their derivatives at $a+\omega$ take on values as stated and are existent at other points. This can be done as is seen by solving these equations for D_{21} and D_{22}. Having made one determination of D_{21} and D_{22} regard them as fixed. The problem now is essentially the same as if $d_{21} \neq 0$.

[†] The existence of extremes will be discussed in section 14.

[‡] In the case of the difference equation $u_n(x) = y_n(x-1)$ and in the case of the differential equation $u_n(x) = y_n(x)$.

Let y_3 and y_4 be any two real linearly independent solutions.

$$\left.\begin{aligned} y_3 &= k_{11} y_1 + k_{12} y_2, \\ y_4 &= k_{21} y_1 + k_{22} y_2. \end{aligned}\right\} \tag{28}$$

$$\left.\begin{aligned} \Delta(x) &\equiv u_3(x) \cdot v_4(x+\omega) - u_4(x) \cdot v_3(x+\omega) \\ &= (k_{11} k_{22} - k_{12} k_{21}) \left(u_1(x) v_2(x+\omega) - u_2(x) v_1(x+\omega)\right). \end{aligned}\right\} \tag{29}$$

It results that $\Delta(x)$ changes sign at $\bar{a}$ and that if $0 < \delta' < \varepsilon$, sufficiently small, $\Delta(\bar{a}+\delta') \cdot \Delta(\bar{a}-\delta') < 0$.

Under the assumption that $\lambda_j(\bar{a})$ is a maximum, I shall derive a contradiction to this statement.

Denote by y_5 and y_6, respectively, solutions such that $u_5(\bar{a}-\delta') = 0$ and $u_6(\bar{a}+\delta') = 0$ and such that at $\bar{a}-\delta'$ and $\bar{a}+\delta'$, respectively, the forward slopes of u_5 and u_6 are the same as that of u_1 at $\bar{a}$.

In the case of the differential equation let $c = \bar{a}$ and in the case of the difference equation let c be that integer, $c < \bar{a} - 1 \leqq c + 1$. Form equation (18) [(18′)] of section 7 for this point c. There exist solutions $\bar{\bar{y}}_1$, $\bar{\bar{y}}_5$ and $\bar{\bar{y}}_6$ such that

$$\begin{aligned} v_1(\bar{a}+\omega) &= \bar{\bar{y}}_1(\bar{a}+\omega), \\ v_5(\bar{a}+\omega-\delta') &= \bar{\bar{y}}_5(\bar{a}+\omega-\delta'), \\ v_6(\bar{a}+\omega+\delta') &= \bar{\bar{y}}_6(\bar{a}+\omega+\delta'). \end{aligned}$$

A root of $\bar{\bar{y}}_1$ lies at $\bar{a}+\omega$. If δ' is taken sufficiently small roots of $\bar{\bar{y}}_5$ and $\bar{\bar{y}}_6$ can be made as close to $\bar{a}+\omega$ as we please. Due to the maximum, however, the situation will be somewhat as here illustrated:

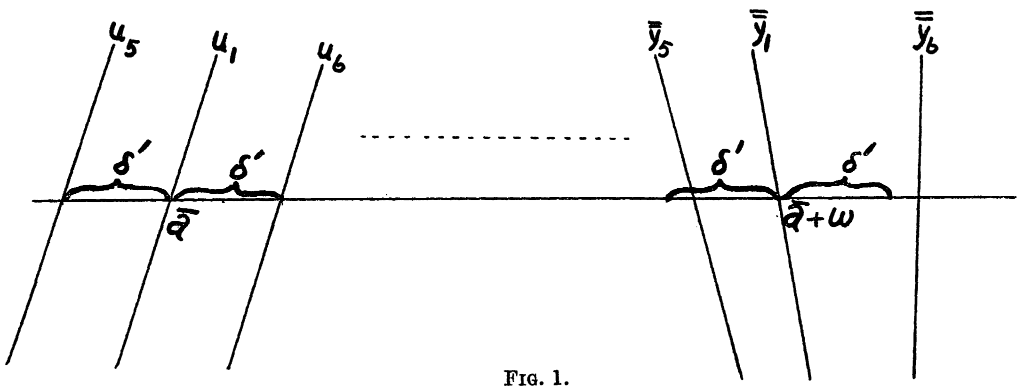

FIG. 1.

We conclude immediately

$$u_5(\bar{a}\pm\delta') v_6(\bar{a}+\omega\pm\delta') - u_6(\bar{a}\pm\delta') \cdot v_6(\bar{a}+\omega\pm\delta') \geqq 0. \tag{30}$$

One can readily assure himself that another arrangement of the figure will not alter this conclusion. Refer now to (29) and we have that $\Delta(\bar{a}+\delta')\cdot\Delta(\bar{a}-\delta)\geqq 0$, the desired contradiction.

§ 11. Suppose that When $\lambda=\Lambda$, $\rho_1=\rho_2$ and that y_7 is a Solution of (21), [(21′)] Satisfying (25) [(25′)] or (26) [(26′)] Such that u_7 Has a Root at $\bar{a}$;* To Prove that Λ is Necessarily an Extreme of a λ_j, the Term Extreme Being Used as Previously.

Under the assumption that $\rho_1=\rho_2$ there exists one solution of (21), y_7 such that $\rho u_7(x)=v_7(x+\omega)$ and a second lineary independent solution y_8 such that $u_8(x+\omega)=C_{21}u_7(x)+\rho u_8(x)$.

$$D(x)=u_7(x)v_8(x+\omega)-u_8(x)v_7(x+\omega)=C_{21}\rho^2[u_7(x)]^2,$$

which obviously never changes sign and does not vanish unless $u_7(x)=0$. We know $u_7(x)$ actually changes sign at $\bar{a}$ and hence when $\delta'>0$ is small enough $D(\bar{a}+\delta')\cdot D(\bar{a}-\delta')>0$.

Assume that Λ is not an extreme of a λ_j. Refer to the determinant (30) discussed in previous section. The corresponding figure now is

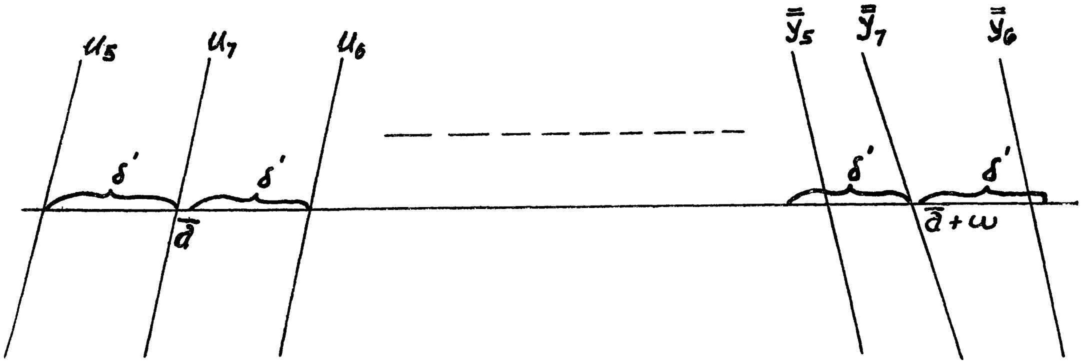

Fig. 2.

from which one concludes that D which is a constant times

$$(u_3(x)v_4(x+\omega)-u_4(x)v_3(x+\omega))$$

does not satisfy the relation $D(\bar{a}+\delta')\cdot D(\bar{a}-\delta')>0$. This is a contradiction and proves the theorem.

§ 12. λ_j a Constant.

Suppose λ_j a constant over some interval $x'\leqq x\leqq x''$. Then by the proof of the previous section λ_j is a value l or a value l'; and, where $\bar{a}$ is any point of the interval $x' \leqq x \leqq x''$, a solution satisfying $u_n(\bar{a}) = 0$ is a solution of the system consisting of (21) [(21′)] and (25) [(25′)] or of (21) [(21′)] and (26) [(26′)]. But for definite points $\bar{a}$ sufficiently close together corresponding solutions satisfying $u_n(\bar{a}) = 0$ are linearly independent. Hence, when $\lambda = \lambda_j$ all solutions of (21) [(21′)] satisfy one of these systems. λ_j is then a constant over the whole interval of its definition.

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* In the case of the difference equation $\bar{a}>a+v$ and in the case of the differential equation $\bar{a}>a+\delta$, where δ is as in section 9 (b).

Conversely, if all solutions of (21) [(21′)] satisfy (25) [(25′)] or (26) [(26′)], the corresponding value of λ is necessarily a value λ_j. For, if $\bar{a}$ is any point on the interval $a \leqq x \leqq a + \omega$, those solutions satisfying $u_n(\bar{a}) = 0$ satisfy $v_n(\bar{a} + \omega) = 0$ also.

§ 13. When λ_j is Not a Constant To Prove All the Maxima of λ_j Equal among Themselves and All the Minima Equal among Themselves.

Let $\lambda_j(c)$ not be an extreme. Suppose that $\lambda_j(\bar{x})$ has two minima less than $\lambda_j(c)$. Denote these by $\bar{l}_{j-1}$ and $\bar{\bar{l}}_{j-1}$, $\bar{\bar{l}}_{j-1} < \bar{l}_{j-1}$. Let $\bar{x}$ decrease from c, λ_j can never get less than l_{j-1} for as soon as it equals $\bar{l}_{j-1}$ suppose at $\bar{c}$ then there exists a solution not identically zero of (21), [(21′)] satisfying (25) [(25′)] or (26) [(26′)] which we denote by y_7 where $u_7(c) = 0$.

Let y_9 be a solution, not identically zero, satisfying $u_9(c) = 0$. Consider $v_7(x+\omega)\, u_9(x) - v_9(x+\omega)\, u_7(x)$. This is a constant times $D(x)$ of section 11. Consequently, it vanishes only when $u_7(x)$ vanishes. But it vanishes at $\bar{c}$ a root of $u_9(x)$. Then by section 11 $\bar{c}$ being a root of $u_7(x)$ is an extreme of λ_j. From the nature of the case it is a minimum; that is, λ_j begins to increase again and the existence of $\bar{\bar{l}}_{j-1}$ is impossible.

§ 14. The Existence of Extremes of the λ_j's.

(a) *The Difference Equation.*

Suppose $\lambda_j(\bar{x})$ $j \geqq 1$ not a constant. Let $\lambda = \lambda_j(a+1)$ and let y_j be a particular solution, not identically zero, satisfying the relation $y_j(a) = 0$. y_j has has at least one node on the interval $a < x \leqq a + \omega$. Let a_1 be the first such root to the right of a.

Theorem: $\lambda_j(a_1+1) = \lambda_j(a+1)$ *and* λ_j *respectively increases or decreases from* a_1 *as* λ_j *increases or decreases from* a.

Proof: Let $\lambda = \lambda_j(a+1)$. It can be immediately shown as was done in section 10 that $\rho u_j(x) \equiv v_j(x+\omega)$, $\rho \neq 0$ a constant. Consequently, $\lambda_j(a_1+1) = \lambda_j(a+1)$. Next let $0 < \bar{x} - a < \delta$, where δ is small, and let $y_{\bar{x}}$ be a solution such that $y_{\bar{x}}(\bar{x}) = 0$ and $y'_{\bar{x}}(\bar{x}) = y'_j(a)$, where the accent denotes the forward derivative. Suppose λ_j increases from a and let $\lambda = \lambda_j(a+1)$. As the nodes of y_j and $y_{\bar{x}}$ separate each other the first node of $y_{\bar{x}}$ to the right of $\bar{x}$ is to the right of a_1. Let λ increase to $\lambda_j(\bar{x}+1)$ and this node moves still farther to the right. Suppose that when $\lambda=\lambda_j(\bar{x}+1)$ it lies at $\bar{a}_1$ and that $\bar{a}_1-a_1=\eta$. η is arbitrarily small, δ being at our disposal. Now, as in the case of $a+1$ and a_1+1, $\lambda_j(\bar{a}_1+1)=(\lambda_j(\bar{x}+1)$. But $\lambda_j(\bar{x}+1)>\lambda_j(a+1)$. Hence, $\lambda_j(\bar{a}_1+1)>\lambda_j(a_1+1)$ which is what we desired to prove. The following theorem results:

$\lambda_j(\bar{x})$, $j \geqq 1$ *has always at least one minimum and at least one maximum.*

As has been remarked these extremes are the roots of the solutions, y_n, not identically zero, satisfying $u_n(x) \equiv \pm v_n(x+\omega)$. The exact number of these roots on $a+1<x \leqq a+\omega$ will be discussed in the theorem of oscillation, section 17.

(b) *The Differential Equation.*

Only obvious formal changes of the treatment given under (a) are necessary.

$\lambda_j(\bar{x})$, $j \geqq 1$ *has always at least one minimum and at least one maximum.*

§ 15. TO DISTINGUISH THE TWO CASES.

(a) *The Difference Equation.*

Assume $\lambda_j=\lambda_j(\bar{a})$. There exists a solution, not identically zero, which we shall call y_j satisfying the relations $y_j(\bar{a}-1)=0$, $v_j(\bar{a}+\omega)=0$. Let c be that integer such that $c<\bar{a}-1 \leqq c+1$ and consider (18) formed for this point.

When λ has any value whatever let y_n denote a solution of (21) satisfying the conditions $y_n(c)=y_j(c)$, $y_n(c+1)=y_j(c+1)$. Then according to section 7 (a) when $a+\omega<x \leqq c+\omega$, $v_n(x+\omega)=y_{11}(x+\omega)$. Let $\eta>0$ be very small.

When $\lambda>0$ is sufficiently large $y_n(c+1+\eta)$ and $y_n(a+\omega+1)$ are of the same sign. Moreover, as shown under section 7, $y_{11}(a+\omega+2)$ and $y_{11}(c+\omega)$ are of the same sign; that is, $v_n(a+\omega+2)$ and $v_n(c+\omega)$ are of the same sign. *To determine, if* $y_n(a+\omega+1)$, *that is,* $u_n(a+\omega+2)$ *is of the same or opposite sign to* $v_n(a+\omega+2)$.

$v_n(a+\omega+2)=y_{10}(a+1)$ [section 7] and as $v_n(a+\omega)=K(a)\Delta y_{10}(a)+by_{10}(a)$ and $\bar{v}_n(a+\omega)=y_{10}(a)$, $K(a)v_n(a+\omega+2)=\bar{D}K(a+\omega)\Delta y_n(a+\omega)+Dy_n(a+\omega)$.

CASE α: $\bar{D}=d_{11}+d_{21}(K(a)-b)\neq 0$. Since $\lim\limits_{\lambda=\infty} \dfrac{K(a+\omega)\Delta y_n(a+\omega)}{y_n(a+\omega)}=\infty$ if λ is sufficiently large $v_n(a+\omega+2)$ will have the sign of $\bar{D}K(a+\omega)\Delta y_n(a+\omega)$, that is, of $\bar{D}y_n(c+1+\eta)$.

CASE β: $\overline{D}=0$. $v_n(a+\omega+2)$ has the sign of

$$Dy_n(a+\omega)=[d_{12}+d_{22}(K(a)-b)]y_n(a+\omega),$$

that is of $Dy_n(c+1+\eta)$, for large values of λ.

Now let λ decrease. The number j is the number of times that $v_n(\bar{a}+\omega)$ has already changed sign. *It results that* $u_j(c+2+\eta)=y_j(c+1+\eta)=\pm v_j(c+\omega+2+\eta)$, *and hence*, $\rho=\pm 1$ *as follows:*

CASE α: j even $\overline{D}>0$ $\rho=-1$, $\overline{D}<0$ $\rho=1$;
j odd $\overline{D}>0$ $\rho=1$, $\overline{D}<0$ $\rho=-1$.
CASE β: j even $D>0$ $\rho=-1$, $D<0$ $\rho=1$;
j odd $D>0$ $\rho=1$, $D<0$ $\rho=-1$.

(b) *The Differential Equation.*

Proceed just as under (a). Suppose $\lambda=\lambda_j(\bar{a})$ an extreme of λ_j. Then there exists a solution y_j, not identically zero, satisfying the relations $u_j(\bar{a})=y_j(\bar{a})=0$ and $v_j(\bar{a}+\omega)=0$. Let y_n be a solution when λ has any value satisfying the relations $y_n(\bar{a})=y_j(\bar{a})$ and $y'_n(\bar{a})=y'_j(\bar{a})$. Then by section 7 when $a+\omega\leqq x\leqq\bar{a}+\omega$, $v_n(x+\omega)=y_{11}(x+\omega)$. Moreover, when $\lambda>0$ is sufficiently large and $\bar{\delta}>0$ small, $y_n(\bar{a}+\bar{\delta})$ and $y'_n(a+\omega)$ have the same sign, and $y_{11}(\bar{a}+\omega)$ and $y_{11}(a+\omega+\bar{\delta})$ are of the same sign; that is, $v_n(a+\omega)$ and $v_n(\bar{a}+\omega)$ are of the same sign. To determine then if $u_n(\bar{a}+\bar{\delta})$ and $v_n(\bar{a}+\omega+\delta)$ are of the same sign, is equivalent to determining if $y'_n(a+\omega)$ and $v_n(a+\omega)$ are of the same sign. This can be immediately done as $v_n(a+\omega)=d_{21}K(a+\omega)y'_n(a+\omega)+d_{22}y_n(a+\omega)$ and $\dfrac{K(a+\omega)y'_n(a+\omega)}{y_n(a+\omega)}\to\infty$ as $\lambda\to\infty$.

CASE (α) $d_{21}\neq 0$: $v_n(a+\omega)$ has the same or opposite sign as $y'_n(a+\omega)$ as $d_{21}\gtrless 0$.

CASE (β) $d_{21}=0$: $v_n(a+\omega)$ has the same or opposite sign as $y'_n(a+\omega)$ as $d_{22}\gtrless 0$.

Completing the reasoning as under (a), we have

CASE α: j even $d_{21}>0$ $\rho=-1$, $d_{21}<0$ $\rho=1$;
j odd $d_{21}>0$ $\rho=1$, $d_{21}<0$ $\rho=-1$.
CASE β: j even $d_{22}>0$ $\rho=-1$, $d_{22}<0$ $\rho=1$;
j odd $d_{22}>0$ $\rho=1$, $d_{22}<0$ $\rho=-1$.

§ 16. SOLUTIONS WITHOUT ROOTS.

(a) *The Difference Equation.*

We desire now to study the existence of values l and l', where the corresponding solutions of (21) have no node on $a+1<x<a=\omega$.

Let $\lambda > 0$ be very large and let y_1 and y_2 be two solutions determined by the conditions

$$y_1(a) = 0, \qquad y_1(a+1) = \frac{1}{K(a)},$$
$$y_2(a) = \frac{1}{K(a)}, \quad y_2(a+1) = 0.$$

Then (see section 2),

$$v_1(a+\omega) = a_{11}u_1(a) + a_{12}u_2(a) = a_{11}\bar{u}_1(a+1) + a_{12}\bar{u}_2(a+1) = (a_{11} - a_{12})\left(1 - \frac{b}{K(a)}\right),$$
$$v_1(a+\omega+1) = \bar{v}_1(a+\omega) = a_{11}u_1(a+1) + a_{12}u_2(a+1) = \frac{a_{12}}{K(a)}.$$

Hence,

$$a_{11} = v_1(a+\omega) + (K(a) - b)\bar{v}_1(a+\omega).$$

Similarly,

$$a_{22} = K(a)\bar{v}_2(a+\omega).$$

The characteristic equation is

$$\rho^2 - (a_{11} + a_{22})\rho + 1 = 0.$$

$$a_{11} + a_{22} = \bar{D}K(a+\omega)\Delta y_1(a+\omega) + Dy_1(a+\omega) + K(a)d_{21}K(a+\omega)\Delta y_2(a+\omega) + d_{22}K(a)y_2(a+\omega).$$

CASE (α): $\bar{D} \neq 0$.

If $\lambda > 0$ is sufficiently large, the sign of $a_{11} + a_{22}$ is that of $\bar{D}$. For, write the difference equation in the form

$$K(i+1)\Delta y(i+1) = K(i)\Delta y(i) + G(i)y(i+1),$$

and the facts, that $y_2(a+1) = 0$ and $y_1(a+1) = \frac{1}{K(a)}$, and that G becomes infinite with λ shows that by taking λ large enough we can make

$$\frac{K(a+1)\,\Delta y_1(a+1)}{|K(a+1)\Delta y_2(a+1)|}$$

as large we please. An easy induction process shows that the same thing is true of $K(a+\omega)\Delta y_1(a+\omega)$ and $|K(a+\omega)\Delta y_2(a+\omega)|$. Moreover, if y_n is any particular solution whatever, not identically zero, with fixed values at a and $a+1$, $K(a+\omega)\Delta y_n(a+\omega)/y_n(a+\omega)$ becomes positively infinite with λ. Hence, $\bar{D}K(a+\omega)\Delta y_1(a+\omega)$ will determine the sign of $a_{11} + a_{22}$. In addition when λ is large enough and $i \geqq a$, $K(i)\Delta y_n(i)$ increases with i, and, consequently, is greater than zero at $a+\omega$. We conclude as desired that the sign of $a_{11} + a_{22}$ is that of $d_{11} + d_{21}K(a)$ when λ is sufficiently large.

CASE (β). $\bar{D} = 0$.

As $K(a+\omega)\Delta y_2(a+\omega)/y_2(a+\omega)$ becomes infinite with λ and as $\Delta y_2(a+\omega) < 0$ for sufficiently large values of λ and as from the fact that $d_{11}d_{22} - d_{12}d_{21} = 1$, $d_{21}D = -1$, the sign of $a_{11} + a_{22}$ is the sign of D.

Suppose now that for large values of λ $a_{11}+a_{22}>0$; that is, $\overline{D}>0$ or $D>0$ as we have cases (α) or (β). Denote the roots of the characteristic equation by ρ_1 and ρ_2. If λ is sufficiently large, $a_{11}+a_{22}$ is as large as we please, and hence, $\rho_1>0$, say, is large and $\rho_2>0$ small. Let λ decrease $(a_{11}+a_{22})$ is continuous in λ, and hence, ρ_1 and ρ_2 are continuous in λ. Consequently, $\rho_1=\rho_2=1$ before $\rho_1=\rho_2=-1$; that is, there exists a value l larger than the largest value l'. But by section 15 the maximum value of $\lambda_0(\overline{x})$, if such exists, is a value l'. *Hence, if* λ_0 *has extremes the solutions corresponding to this largest value* l *can have no node when* $a\leqq x\leqq a+\omega$.

There can not also be a value l' such that a corresponding solution has no node when $a\leqq x\leqq a+\omega$ for by the methods of section 15 solutions without roots always correspond definitely either to a value l or to a value l'.

If for large values of λ $a_{11}+a_{12}<0$ the above situation is exactly reversed and if $\lambda_0(\overline{x})$ has extremes there exists a value l' and no value l such that a corresponding solution has no node, $a\leqq x\leqq a+\omega$.

Consider now the case that $a_{11}+a_{22}>0$ for large positive values of λ. *We wish to prove that there can not be two distinct values* l *such that corresponding solutions have no node on* $a+1\leqq x\leqq a+\omega$. I find it here necessary to assume K independent of λ an assumption not made in the preceding. Suppose there were two such values l and l'. Denote corresponding solutions by y and $\overline{y}$ and assume that y and $\overline{y}$ are of the same sign when $a+1<x<a+\omega$. We have from (15)

$$\begin{aligned} K(a+\omega)(y,\overline{y};\ a+\omega,\ a+\omega+1)-K(a)(y,\overline{y};\ a,\ a+1) \\ =\sum_{i=a}^{a+\omega-1}(\overline{G}(i)-G(i))\overline{y}(i+1)y(i+1). \end{aligned} \tag{31}$$

But as y and $\overline{y}$ both satisfy (25),

$$K(a+\omega)(y,\overline{y};\ a+\omega,\ a+\omega+1)=K(a)(y,\overline{y},\ a,\ a+1). \tag{32}$$

(31) *and* (32) *are inconsistent under the hypothesis that* $y(x)$ *and* $\overline{y}(x)$ *preserve the same sign when* $a+1<x<a+\omega$.

If $a_{11}+a_{22}<0$ *for large values of* λ, *we show in exactly the same way that there can not be two values* l' *such that corresponding solutions have no node.*

(b) *The Differential Equation.*

Let y_1 and y_2 be two solutions determined by the conditions

$$y_2(a)=0, \qquad K(a)y_2'(a)=1,$$

$$y_1(a)=\frac{1}{K(a)}, \qquad K(a)y_1'(a)=0.$$

Then
$$v_1(a+\omega)=a_{11}u_1(a)+a_{12}u_2(a)=\frac{a_{11}}{K(a)},$$
$$\bar{v}_2(a+\omega)=a_{21}\bar{u}_1(a)+a_{22}\bar{u}_2(a)=\frac{a_{22}}{K(a)}.$$

Hence,
$$a_{11}+a_{22}=[v_1(a+\omega)+\bar{v}_2(a+\omega)]K(a)=[d_{21}K(a+\omega)y_1'(a+\omega)$$
$$+d_{22}y_1(a+\omega)+d_{11}K(a+\omega)y_2'(a+\omega)+d_{12}y_2'(a+\omega)]K(a).$$

Now let $\lambda=\bar{\lambda}$ be so large that y_1 and y_2 increase throughout the interval $a<x\leqq a+\omega$,
$$\frac{d}{dx}\frac{y_2}{y_1}=\frac{y_1y_2'-y_2y_1'}{y_1^2}=\frac{1}{K(a)K(x)y_1^2},$$
$$\frac{y_2(\alpha)}{y_1(\alpha)}=\frac{1}{K(a)}\int_a^\alpha\frac{dx}{K(x)y_1^2}\quad a\leqq\alpha\leqq a+\omega.$$

Now let λ increase, when $a<x\leqq a+\omega$ $\frac{1}{y_1^2}$ approaches zero as λ becomes infinite and $1/K(x)$ does not increase. Let $K(x)>\zeta$, $a\leqq x\leqq a+\omega$, when $\lambda\geqq\bar{\lambda}$.

Then $\int_a^\alpha\frac{dx}{K(x)y_1^2}\leqq\frac{1}{\zeta}\int_a^\alpha\frac{dx}{y_1^2}$. But $\frac{1}{(y_1(x))^2}\leqq 1$, and when $a<x\leqq a+\omega$ it approaches zero. It results that
$$\lim_{\lambda=\infty}\frac{y_2(\alpha)}{y_1(\alpha)}=\lim_{\lambda=\infty}\frac{1}{K(a)}\int_a^\alpha\frac{dx}{K(x)(y_1(x))^2}=0.$$

This approach is uniform in α since $\int_a^{a+\omega}\frac{dx}{K(x)(y_1(x))^2}>\int_a^\alpha\frac{dx}{K(x)(y_1(x))^2}$ when $a+\omega<\alpha$.

Now, moreover, from the differential equation
$$\frac{K(a+\omega)y_2'(a+\omega)}{K(a+\omega)y_1'(a+\omega)}=\frac{\int_a^{a+\omega}G(x)y_2(x)\,dx+1}{\int_a^{a+\omega}G(x)y_1(x)\,dx}.$$

Hence, $\frac{y_2'(a+\omega)}{y_1'(a+\omega)}\leqq\int_a^{a+\omega}\frac{y_2(x)}{y_1(x)}dx+\frac{1}{\int_a^{a+\omega}G(x)y_1(x)\,dx}$ when $\lambda\geqq\bar{\lambda}$. Consequently, $\lim_{\lambda=\infty}\frac{y_2'(a+\omega)}{y_1'(a+\omega)}=0$. Hence, since $\lim_{\lambda=\infty}\frac{K(a+\omega)\,y_k'(a+\omega)}{y_k(a+\omega)}=\infty$; $k=1, 2$, the sign of $a_{11}+a_{22}$ is that of d_{21} if $d_{21}\neq 0$. If $d_{21}=0$ $d_{22}d_{11}=1$ and the sign of $a_{11}+a_{22}$ is that of d_{22}.

The reasoning is now continued exactly as under (a) with the same result. In order to give the uniqueness proof, using the analogous formula to (31) however, *it is necessary to assume K independent of λ*. There has been no call for this assumption earlier in the paper.

§ 17. THE OSCILLATION THEOREM FOR THE DIFFERENCE EQUATION.*

A solution y_p, *where* $\lambda_{p-1}(a+2) \geqq l_p \geqq \lambda_p(a+2)$, *has* $p-1$, p *or* $p+1$ *nodes on the interval* $a+1 < x < a+\omega$ *according to the following table.*

$$\text{Let } \frac{K(a+\omega)\,\Delta y_p(a+\omega)}{y_p(a+\omega)} = R_p,$$

1] $y_p(a+\omega) \neq 0$,

$\overline{D} > 0$ and $R_p > -\frac{D}{\overline{D}}$,

$p=2m$, p nodes,
$p=2m+1$, $p+1$ nodes;

$\overline{D} < 0$ and $R_p > -\frac{D}{\overline{D}}$,

$p=2m$, $p+1$ nodes,
$p=2m+1$, p nodes;

$\overline{D} > 0$ and $R_p < -\frac{D}{\overline{D}}$,

$p=2m$, $p-1$ nodes,
$p=2m+1$, p nodes;

$\overline{D} < 0$ and $R_p < -\frac{D}{\overline{D}}$,

$p=2m$, p nodes,
$p=2m+1$, $p-1$ nodes;

$\overline{D} = 0$ but $D > 0$,

$p=2m$, p nodes,
$p=2m+1$, $p-1$ nodes;

$\overline{D} = 0$ but $D < 0$,

$p=2m$, $p-1$ nodes,
$p=2m+1$, p nodes.

2] $y_p(a+\omega) = 0$,

$d_{11} + d_{21}K(a) > 0$,

$p=2m$, $p-1$ nodes,
$p=2m+1$, p nodes;

$d_{11} + d_{21}K(a) < 0$,

$p=2m$, p nodes,
$p=2m+1$, $p-1$ nodes.

3] $\overline{D}K(a+\omega)\,\Delta y_p(a+\omega) + D\,y_p(a+\omega) = 0$.

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[*] The corresponding theorem for the differential equation is not given. It is that given by Birkhoff: *Amer. Math. Soc. Trans.*, Vol. X, p. 269.

This includes all cases not included under 1] and 2]. y_p has p nodes on $a+1<x<a+\omega$ except when $l_p=\lambda_{p-1}(a+2)$ when it has only $p-1$.

PROOF: When λ equals any $\lambda_j(a+2)$, all solutions satisfying $y_n(a+1)=0$ satisfy $\bar{D}K(a+\omega)\Delta y_n(a+\omega)+Dy_n(a+\omega)=0$ also; for $v_n(a+\omega+2)=\bar{y}_n(a+\omega+2)=0$ and $\bar{\bar{y}}_n(a+\omega+2)=\bar{D}K(a+\omega)\Delta y_n(a+\omega)+Dy_n(a+\omega)$. [See section 7].

Now denote by y_n that solution of (21) satisfying the relations

$$K(a+\omega)\Delta y_n(a+\omega)=D \text{ and } y_n(a+\omega)=-\bar{D}.$$

When $\lambda=l_p$ y has on the interval $a+1<x<a+\omega$, p nodes, or $p-1$ nodes an additional node lying at a. But nodes of two solutions coincide or separate each other, and hence, y_p has on the interval $a+1<x<a+\omega$ either $p-1$, p or $p+1$ nodes.

Now suppose $\bar{D}\neq 0$ and $R_p>-\frac{D}{\bar{D}}=R_n$, then y_p has at least as many nodes on the interval $a+1<x<a+\omega$ as y_n has, that is, p or $p+1$. This is a well-known theorem and is assumed. Moreover, from (25)

$$y_p(a+1)=y_p(a+\omega)\bar{D}\left[R_p+\frac{D}{\bar{D}}\right]$$

and consequently $y_p(a+1)$ and $y_p(a+\omega)$ have the same or opposite sign as $\bar{D}\gtrless 0$. The table follows immediately.

If $R_p<-\frac{D}{\bar{D}}$, y_p has no more nodes than y_n and the table follows as in the other case. Similarly, the table follows immediately from (25) if $\bar{D}=0$.

If $y_p(a+\omega)=0$ but $\bar{D}\neq 0$ an application of the theorem that the nodes of y_p and y_n separate each other shows that y_p has no more nodes than y_n has on the interval $a+1<x<a+\omega$, that is, not more than p. But from (25) we have the relation $y_p(a+1)=\bar{D}y_p(a+\omega+1)$. Hence, the number of nodes of y_p on $a+1<x<a+\omega$ is odd or even as $\bar{D}\gtrless 0$. The table follows.

The statement 3] is immediate and is assumed in what has preceded.

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